Equatorial Reflection Sundials[1]

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This article describes a system of reflection dials, installed on the north facing wall of a house in S. Maria Maggiore (VB) Italy, that exhibit the typical layout of equatorial dials. The design method is then explained applying different computational approaches.

Val Vigezzo in Italy, the place where the XX National Gnomonic Conference will be held in October 2015, includes several sundials. Among them a group of four sundials is worthy of attention because they are technically and artistically well designed and because of the technical solution adopted - both original and fascinating.

Being so innovative, it can be difficult to understand how they were designed and this is the reason for this article.

The four sundials are installed in S. Maria Maggiore (VB – Italy) and can be found on Sundial Atlas (http://www.sundialatlas.eu/) as IT005738, IT011411, IT011412 and IT011413.

They are symmetrically placed on the northern wall of the house, two on the left and two on the right corner (fig. 1). The mirrors, that can hardly be seen in the photo, are placed in the center of each circular sector.

In order to understand the behavior of these dials we have to remember that a reflection dial reflects a half of the celestial sphere to the wall and hour lines and day lines are consequently modified.

Fig. 1 – Reflection sundials at Garavaglia House in S. Maria Maggiore (VB – Italy) – photo by G. de Donà

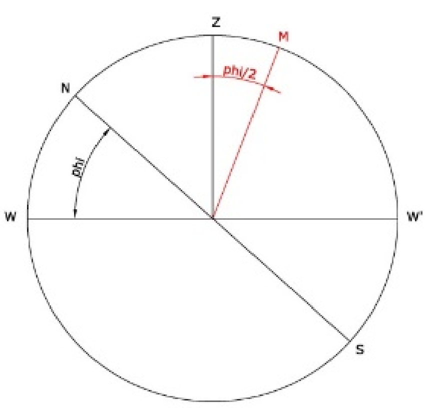

Fig. 2 –Projection of the north pole to the wall plane (summer)

By varying the orientation of the mirror it is possible to obtain a layout that is completely different from what we would expect on a wall with that declination.[2]

It can be noted that the four sundials look like equatorial dials *i.e.* they seem to lie on a plane that is parallel to the equatorial plane. If they are carefully analyzed it looks evident that the upper ones are really the left and the right section of a north-equatorial sundial while the lower ones are really the left and the right sections of a south-equatorial sundial.

In order to get these layouts, the mirrors must be placed in such a way that the upper and the lower celestial hemispheres are respectively reflected at right angles to the wall surface (as it

[1] This paper was originally published in the Italian magazine “Orologi Solari” n. 7, April 2015.

[2] This concept was first stated and illustrated by Tonino Tasselli in [2].

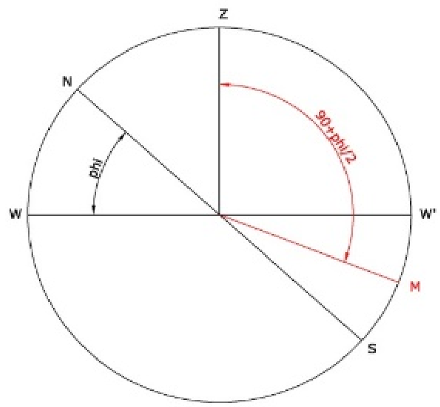

Fig. 3 –Projection of the south pole to the wall plane (winter)

actually happens for a true equatorial sundial).

As an example, if the wall is facing exactly north (declination is 180°) this requirement is satisfied when the poles N and S are projected into W' by means of a mirror M with inclination I_m with respect to the horizon:

$I_m = \varphi/2$ for the north-equatorial dial

$I_m = 90^\circ + \varphi/2$ for the south-equatorial dial

where φ is the latitude (see figs. 2 and 3 that show the meridian plane where NS is the earth axis, Z is the zenith, W is the wall pole and M is the mirror pole).

The place where the four dials are installed has latitude 46.1337° and so mirrors must be inclined 23.067° (the upper summer pair) and 113.067° (the lower winter pair). Of course, as it happens for any equatorial dials, they will not work at the equinoxes when the reflected light ray will be parallel to the wall and the spotlight will be at an infinite distance.

Actually the wall is not exactly facing north and has declination 178.8° east (value obtained from the sheet IT005738); any consideration must then be taken on the celestial sphere and not on the meridian plane as in figs. 2 and 3.

Fig. 4 shows the celestial sphere including the north pole N, the zenith Z of the place with latitude φ, the pole W of the wall (considered vertical with declination D_w) and the opposite direction W' where the mirror M must reflect the pole N at right angles. The mirror M is oriented with declination D_m and inclination I_m.

Analyzing the spherical triangles NZW' and MZW' the following relationships can be obtained:

$$\cos NW' = \cos\varphi \cos D_w$$

$$NM = MW' = NW'/2$$

$$\sin W' = \cos\varphi \sin D_w / \sin NW'$$

$$\tan MZW' = \sin W' \tan MW'$$

$$\sin MZ = \sin W' \sin MW' / \sin MZW'$$

$$D_m = -\left(MZW' + 180^\circ - D_w\right)$$

$$I_m = MZ$$

With latitude φ = 46.1337° N and wall declination = 178.8° E we obtain:

$D_m = 3.905^\circ W$ $I_m = 23.1^\circ$

In a similar way for winter dials, as shown in fig. 5, analyzing the spherical triangles SZW' and MZW', the following relationships are obtained:

$$\cos SW' = -\cos\varphi \cos D_w$$

$$SM = MW' = SW'/2$$

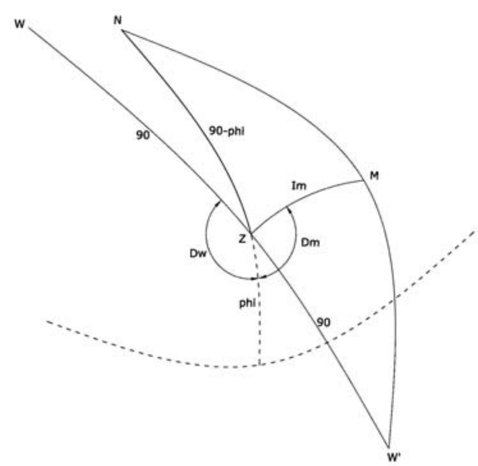

Fig. 4 –Projection of the north pole to the declining wall (summer)

$$\sin W' = \cos\varphi \sin D_w / \sin SW'$$
$$\tan MZW' = \sin W' \tan MW'$$
$$\sin MZ = \sin W' \sin MW' / \sin MZW'$$
$$D_m = -\left(180^\circ - D_w - MZW'\right)$$
$$I_m = MZ$$

With latitude 46.1337° N and wall declination 178.8° E we obtain:

$D_m = 0.709^\circ W \qquad I_m = 113.068^\circ$

Because of the small deviation from true north these new values are very close to the ones previously obtained for a north facing wall.

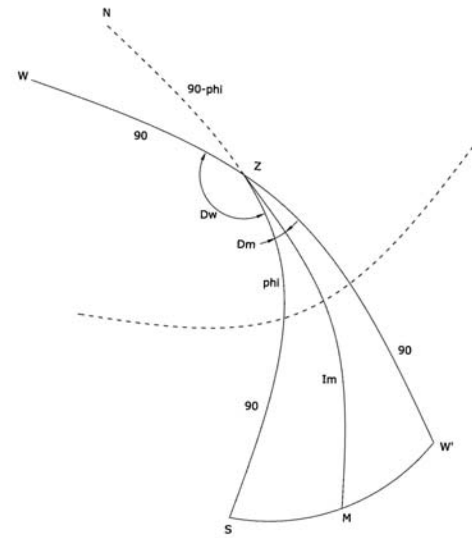

Fig. 5 –Projection of the south pole to the declining wall (winter)

When the deviation is larger, the large value of the mirror declination produces a rotation of the layout and of the horizon line: see as an example figs. 6 and 7 where the sundials are for a wall declining 150° East.[3]

Mirror orientation is in this case:

$D_m = 70.91^\circ W \qquad I_m = 36.27^\circ$ for the summer dials

$D_m = 17.78^\circ W \qquad I_m = 113.77^\circ$ for the winter dials

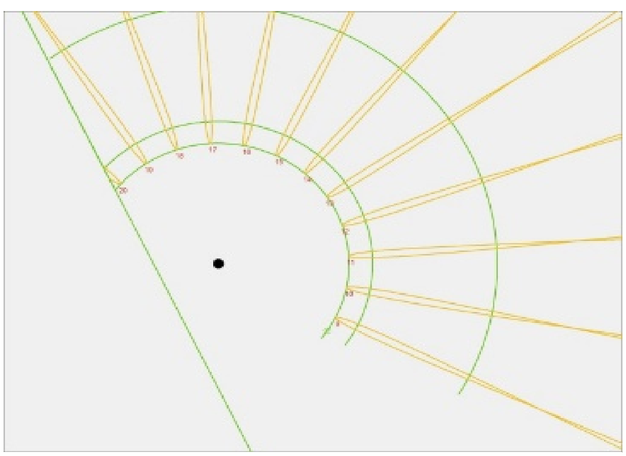
Fig. 6 – Wall 150° east : summer dial

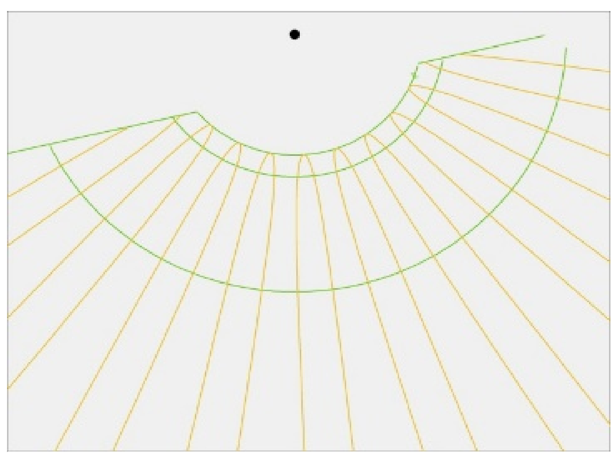
Fig. 7 – Wall 150° east : winter dial

The same results can be obtained using vector algebra instead of spherical trigonometry.

At the Gnomonic Conference in Lignano [1] and in Chianciano [2], the Italian dialist Tonino Tasselli showed how vector algebra can be profitably used for the design of both classic and reflection sundials.

As explained in [2] reflection by the mirror of the incident ray $\vec{a}$ can be described by the Householder H matrix:

[3] Sundials in figs. 6 and 7 can be designed with the program "Orologi Solari" rev. 28.5 or later that can be downloaded from http://www.sundials.eu/download/download.html

$$\vec{b} = H \cdot \vec{a} \qquad H = \begin{pmatrix} (1-2n_x^2) & -2n_x n_y & -2n_x n_z \\ -2n_x n_y & (1-2n_y^2) & -2n_y n_z \\ -2n_x n_z & -2n_y n_z & (1-2n_z^2) \end{pmatrix},$$

where $\vec{n} = n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$ is the unitary vector orthogonal to the mirror.

The northern equatorial sundial is obtained by requiring that ray $\vec{a}$ coming from the north pole is reflected into ray $\vec{b}$ orthogonal to the wall.

In an orthogonal reference system where *xy* is the local horizontal plane, $\hat{x}$ is oriented to south, $\hat{y}$ to east and $\hat{z}$ to the zenith, such vectors $\vec{a}$ and $\vec{b}$ are

$$\vec{a} = \hat{x}\cos\varphi - \hat{z}\sin\varphi \qquad \vec{b} = -\hat{x}\cos D_w + \hat{y}\sin D_w$$

Solving for n_z, n_x, and n_y we obtain:[4]

$$n_z = \pm\sqrt{\frac{\sin^2\varphi}{2+2\cos\varphi\cos D_w}} \qquad n_x = -\frac{\cos\varphi + \cos D_w}{\sin\varphi} n_z \qquad n_y = \frac{-\sin D_w}{2n_x\cos\varphi - 2n_z\sin\varphi}$$

For the sundial in S. Maria Maggiore we get:

$n_z = 0.9198 \qquad n_x = 0.3914 \qquad n_y = -0.0267$, from which we obtain:

$$D_m = -\tan^{-1}\left(\frac{n_y}{n_x}\right) = 3.905^\circ \qquad I_m = \tan^{-1}\left(\frac{\sqrt{n_x^2 + n_y^2}}{n_z}\right) = 23.1^\circ$$

In a similar way, when projecting the south pole to the wall we get:

$$\vec{a} = -\hat{x}\cos\varphi + \hat{z}\sin\varphi \qquad \vec{b} = -\hat{x}\cos D_w + \hat{y}\sin D_w$$

$$n_z = \pm\sqrt{\frac{\sin^2\varphi}{2-2\cos\varphi\cos D_w}} \qquad n_x = \frac{\cos D_w - \cos\varphi}{\sin\varphi} n_z \qquad n_y = \frac{-\sin D_w}{-2n_x\cos\varphi + 2n_z\sin\varphi}$$

For the wall in S. Maria Maggiore we get: $n_z = -0.3918 \qquad n_x = 0.92 \qquad n_y = -0.0114$

$$D_m = -\tan^{-1}\left(\frac{n_y}{n_x}\right) = 0.709^\circ \qquad I_m = \tan^{-1}\left(\frac{\sqrt{n_x^2 + n_y^2}}{n_z}\right) = 113.068^\circ$$

References:

[1] T. Tasselli, "*Applicazioni del calcolo vettoriale alla gnomonica*", XIII Italian Gnomonic Conference, Lignano (Ud), 8-9-10/4/2005, Proceedings, pages 162-177.

[2] T. Tasselli, "*Applicazioni del calcolo vettoriale alla gnomonica – riflessione e rifrazione*", XIV Italian Gnomonic Conference, Chianciano (Si), 6-7-8/10/2006, Proceedings, pages 173-181.

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[4] The two solutions resulting from the ± sign correspond to the normal direction to the two faces of the mirror. The correct one must be selected, *e.g.* by choosing the one that satisfies the condition $\vec{b} \cdot \hat{n} > 0$.